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Braid Group Symmetries

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Some invariance transformations of the braid group are used to construct crystallographic and lattice representations. A set of related properties are analyzed.

1. INTRODUCTION

In pure mathematics, the theory of braids has been used as a tool to investigate the theory of knots. In addition, this is well known that in quantum theory in two and three space-time dimensions the statistics of particles and fields are not in general described by representations of the permutation group. It was recognized only recently that the new field statistics encountered in two-dimensional models with quantum kinks may lead to representations of the braid groups (Fröhlich, 1976; Bellisard *et al.*, 1978).

In three-dimensional local quantum theory, which is of considerable interest in condensed matter physics, it is a rather old observation that particle statistics is described by representations of the braid groups (Schulman, 1971; Laidlaw and De Witt-Morette, 1971). Braid statistics also appear to play an important role in systems exhibiting a fractional quantum Hall effect (Fröhlich, 1989).

Furthermore, braid and quantum groups follow from a Yang–Baxter algebra in the limit of $\pm\infty$ for the spectral parameter (de Vega, 1989).

In this paper we show the crystallographic and lattice groups as a result of symmetries of braid groups. Thus, a connection is given between the group of statistics in three-dimensional local quantum models and the pure space symmetry of physical crystals.

The paper is organized as follows. Section 2 presents a short review about braid groups and some concepts related to this subject. Section 3

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presents a construction of crystallographic and lattice groups using an analysis of the symmetries given by the braid group relations, and studies related properties.

Finally, Section 4 presents our conclusions.

2. BRAID GROUPS

Let us consider a rectangle P and two sets of n points $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ placed at equal distances on a pair of opposite sides (L_1 and L_2). Suppose that $A_i \to B_i$ is a one-to-one correspondence between these sets of points. P is assumed to be placed in the space R^3.

We take simple broken lines l_i in R^3 connecting A_i and B_{k_i} such that l_i, l_j are disjoint for $i \neq j$, the projection l_i' of l_i on the plane of P lies inside P, and any straight line parallel to L_1 and L_2 and lying between them intersects l_i at a single point for each i. Finally, we assume that l_i' and l_j' for $i \neq j$ intersect at finite points and that they are all in different levels. Such a configuration is called a braid of nth order. An example of a braid of nth order is given in Figure 1.

Each topologically equivalent class of braids is identified with an element of the group.

When the lines l_i, l_j have no intersections, the braid is called trivial. A general n-braid is obtained from the trivial one by applying successively the operations b_i and/or the inverses b_i^{-1} $(1 \leq i \leq n-1)$. The operations b_i and b_i^{-1} are depicted in Figure 2; for simplicity we avoid the frame P in this description.

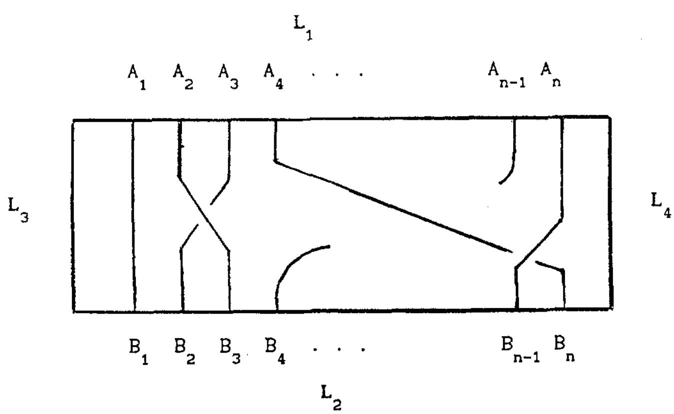

Fig. 1. A braid of nth order (n-braid).

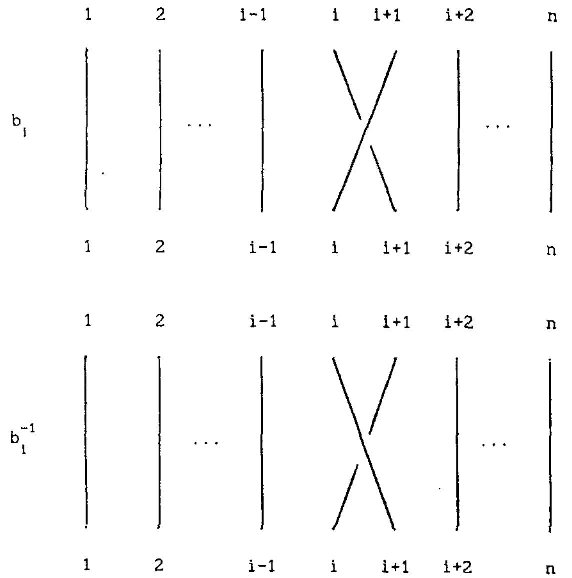

Fig. 2. The elementary operations b_i and b_i^{-1} from the n-braid group.

The operations b_i and b_i^{-1} are the n-braid group generators and they obey the relations

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \\ b_i b_j &= b_j b_i \qquad \text{if} \quad |i-j| \geq 2 \end{aligned} \tag{2.1}$$

It is well known that quantum and braid groups follow from Yang–Baxter algebras (de Vega, 1989).

Consider the nonsingular matrix $R(\theta) = [R_{ab}^{cd}(\theta)]$ as a Yang–Baxter bundle with spectral parameter θ; therefore it satisfies the equation

$$\begin{aligned} &R_{a_2a_1}^{cd}(\theta_1 - \theta_2) R_{a_3c}^{b_1e}(\theta_1 - \theta_3) R_{ed}^{b_2b_3}(\theta_2 - \theta_3) \\ &\quad = R_{a_3a_2}^{mn}(\theta_2 - \theta_3) R_{na_1}^{pb_3}(\theta_1 - \theta_3) R_{mp}^{b_1b_2}(\theta_1 - \theta_2) \end{aligned} \tag{2.2}$$

Moreover, if we only assume that

$$\lim \Theta_{\Rightarrow \pm\infty} R(\theta) = R_{\pm} \tag{2.3}$$

exists and is nonzero, these $R_{\pm}$ matrices can be represented graphically as in Figure 2. This means that we have

$$(R_{i,i+1}^{i,i+1})_+ = b_i \qquad \text{and} \qquad (R_{i,i+1}^{i,i+1})_- = b_i^{-1} \tag{2.4}$$

In addition, it is easy to prove that

$$R_+ R_- = R_- R_+ \qquad \text{or} \qquad b_i^{-1} b_i = b_i b_i^{-1} = 1 \tag{2.5}$$

3. CRYSTALLOGRAPHIC AND LATTICE REPRESENTATIONS

Let us consider an $\mathbf{X}$ element of an n-dimensional Euclidean space R^n,

$$\mathbf{X}=X^i e_i, \qquad i=1, \ldots, n \tag{3.1}$$

where we have used (e_i) as a basis set of R^n and the Einstein convention for the indices.

We can link the components of the vector (3.1) to the two sets of n points that define an n-braid in such a way that X_i corresponds simultaneously to A_i and B_i (see Figure 1). Thus, it is clear that we have a direct realization of the braid group generators

$$b_i\mathbf{X}=b_i^{-1}\mathbf{X}=X^1e_1+\cdots+X^{i+1}e_i+X^ie_{i+1}+\cdots+X^ne_n \tag{3.2}$$

This is because (3.2) satisfies (2.1). This means that we have

$$\begin{aligned} b_ib_j\mathbf{X}&=b_jb_i\mathbf{X} \qquad \text{if} \quad |i-j|\geq 2 \\ b_ib_{i+1}b_i\mathbf{X}&=b_{i+1}b_ib_{i+1}\mathbf{X} \end{aligned} \tag{3.3}$$

with the additional condition

$$b_i\mathbf{X}=b_i^{-1}\mathbf{X} \qquad \text{or} \qquad b_i^2\mathbf{X}=1 \tag{3.4}$$

We remark that (3.4) is a restriction given for the realization (3.2), but it is not needed to define the braid group. Only in the particular case $R_+ = R_-$ is this assumption justified; it is a usual mistake in the literature to consider the relations (2.1) and (3.4) as fundamental ones to define the n-braid group algebra (Bacry, 1989).

The braid realization (3.2) is not the only one. It is possible to get a family of braid realizations with or without the additional restriction given by (3.4). We can consider all of these possibilities as symmetry operations that keep invariant the braid relations (2.1).

Let us consider the operator

$$b_i\mathbf{X}=X^1e_1+\cdots+(\alpha X^{i+1}+\beta)e_i+(\gamma X^i+\delta)e_{i+1}+\cdots+X^ne_n \tag{3.5}$$

with the corresponding inverse

$$b_i^{-1}\mathbf{X}=X^1e_1+\cdots+(X^{i+1}-\delta)/\gamma e_i+(X^i-\beta)/\alpha e_{i+1}+\cdots+X^ne_n \tag{3.6}$$

We can prove that (3.5) and (3.6) satisfy the braid group relations (2.1) if and only if

$$\gamma\beta+\delta=\alpha\delta+\beta \tag{3.7}$$

is true.

Before we proceed, it is clear that (3.5) and (3.6) (for any values of α, β, γ, and δ) generate n linearly independent translations (being a subgroup of the group of motion in an n-dimensional Euclidean space R^n), which means that they are a representation of the n-dimensional crystallographic group. For the particular case given by $\beta=\delta=0$ we have a subgroup of the crystallographic group called the lattice group.

The case $\beta=0$ has been analyzed in the literature (Bacry, 1989) and defines either $\delta=1$ or δ as an ideal of α.

To analyze the internal symmetries of the Yang–Baxter algebras associated to the braid group realization (3.5) and (3.6) we need a matrix representation for b_i and b_i^{-1},

$$b_i = \sum_{\substack{\alpha\neq i\\ \alpha\neq i+1}} E_{\alpha,\alpha} + \beta X_i^{-1} E_{i,i} + \delta X_{i+1}^{-1} E_{i+1,i+1} + \alpha E_{i,i+1} + \gamma E_{i+1,i} \tag{3.8}$$

$$\begin{aligned} b_i^{-1} = {} & \sum_{\substack{\alpha\neq i\\ \alpha\neq i+1}} E_{\alpha,\alpha} - \delta(\gamma X_i)^{-1} E_{i,i} - \beta(\alpha X_{i+1})^{-1} E_{i+1,i+1} \\ & + \alpha^{-1} E_{i,i+1} + \gamma^{-1} E_{i,i+1} \end{aligned} \tag{3.9}$$

where $E_{\alpha,\beta}=\delta_{i\alpha}\delta_{j\beta}$, $X_i=\alpha X_{i+1}$, and we have only two possibilities, either $\beta=0$ or $\delta=0$.

Let us consider the group of matrices $\mathcal{G}$ with the property

$$[g\otimes g, R(\theta)]=0, \qquad \forall g\in\mathcal{G} \tag{3.10}$$

The connection between the θ dependence of $R(\theta)$ and the symmetry group $\mathcal{G}$ is well known (de Vega, 1989).

From (3.10) in the limit $\theta \to \pm\infty$, we can deduce

$$\begin{aligned} g\otimes g b_i &= b_i g\otimes g \\ g\otimes g b_i^{-1} &= b_i^{-1} g\otimes g \end{aligned} \tag{3.11}$$

In our case, for $\beta=0$ and the matrix representation given by (3.8) and (3.9), we obtain $\gamma[g\otimes g]_{i,i+1}=\alpha[g\otimes g]_{i+1,i}$ as the only condition over the internal symmetry group $\mathcal{G}$. Therefore, it is clear the connection between $\mathcal{G}$ and the form given to the crystallographic group in (3.5) and (3.6).

4. CONCLUSIONS

In this paper we have shown the general way to obtain the crystallographic (and lattice) transformations as a family of symmetries of the relations defining a braid group. Furthermore, we present the connection between the internal symmetry of the Yang–Baxter equation which generates, in the $\pm\infty$ limit, the braid group and the explicit form for the crystallographic (and lattice) transformation.

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